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Solutions of Luikov equations of heat and mass transfer in capillary porous bodies through matrix calculus: a new approach

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Abstract

In this paper an eigenvalue analysis approach is employed to obtain the solutions of the Luikov system of linear partial differential equations addressed to the most general type of boundary conditions. The Luikov equations provide a well established model for the analysis of various simultaneous heat and mass diffusion problems in capillary porous bodies. However, analytical methods to achieve a complete and satisfactory solution of these equations is still lacking in the literature because of noninclusion of the existence of a countable number of complex roots in almost all the solutions. A specific example on contact drying of a moist porous sheet with uniform initial temperature and moisture distribution is considered. The influence of the complex roots on the dimensionless temperature, moisture content, and the local rate of drying is demonstrated. A set of benchmark results is obtained for reference purposes.

Nomenclature

a_m diffusion coefficient of moisture in the material [m^2 s^{-1}]
a_q thermal diffusion coefficient [m^2 s^{-1}]
c_m specific moisture capacity [kg (moisture) (kg (dry body)$^{-1}$ °M^{-1}]
c_q specific heat capacity [J kg^{-1} K^{-1}]
R thickness of the layer of the moist material [m]
t time variable [s]
v temperature [°C]
w moisture potential [°M]
x^* space variable [m].

Greek symbols

α_m convective moisture transfer coefficient [kg m^{-2} s^{-1} °M^{-1}]
α_q convective heat transfer coefficient [W m^{-2} K^{-1}]
γ_0 density of the dry portion of the moist body [kg m^{-3}]
δ thermogradient coefficient [°M K^{-1}]
ε ratio of vapour diffusion coefficient to that of total moisture diffusion or evaporation number
λ_m moisture conductivity coefficient [kg m^{-1} s^{-1} °M^{-1}]
λ_q thermal conductivity coefficient [J m^{-1} s^{-1} K^{-1}]
ρ heat of phase change [J kg^{-1}].

Subscripts

s ambient
p equilibrium with ambient
0 initial condition.

1. Introduction

The importance of heat and mass transfer in capillary porous materials has increased in the last few decades due to its wide industrial as well as research applications. Besides its various terrestrial applications (e.g., ground water pollution, heat transfer and storage of solar energy in the ground, packed columns in chemical industries, drying and multiphase flow under non-isothermal conditions), it is being widely used in space research

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especially in devices for liquid and energy transfer (heat pipes, heat exchangers, insulation of highly conducting wire nets, etc.) due to the fact that the performance of capillary porous materials does not depend on a gravitational field. As a result there is a continuously increasing research activity in the field of heat and mass transfer in capillary porous media. But because of the complexities of the mechanism involved in the transport processes through irregular void configuration in porous bodies, only limited success has been achieved in modelling the process.

In order to describe the history of moisture transfer in capillary porous bodies, the dependence on the relevant material characteristics, such as, the topology of solid matrix, interface phenomena among solid, liquid, gaseous vapour and air and liquid–vapour equilibria must be taken into account. The phenomena appropriate for moisture, pressure, and enthalpy distributions are coupled. A diffusion theory with a linear or non-linear coefficient of diffusivity will not serve the purpose for description of the behaviour of mass transfer in a capillary porous body. The transport of associated matter of all phases and the transfer of enthalpy must be considered simultaneously. Therefore, a general mathematical model for multi-phase moisture transfer in capillary porous bodies must be formulated. Various theoretical models have been proposed [1–8] in the past. Most of these models have not been able to predict the drying rate and the distribution of temperature and moisture potentials for both hygroscopic and non-hygroscopic materials over a wide range of boundary conditions and drying regimes. Luikov [2] developed a uniquely different model for simultaneous heat and moisture transfer in capillary porous materials, which is based on non-equilibrium thermodynamics. This model is applicable for both hygroscopic and non-hygroscopic materials. It is interesting to note that the model proposed by Krischer [1] is identical to that of Luikov and the model proposed by De-Vries [3] is similar to that of Luikov [2]. However, the Luikov system of equations is a non-linear system because the transport coefficients and the thermodynamic properties (specific heat, thermogradient coefficient, etc.) are functions of either moisture content or temperature or both. In order to make the system more mathematically tractable Luikov and Mikhailov [9] suggested that if calculation of time dependent heat and mass transfer is carried out by zones, in each of which the transport coefficients are taken as constant (average value for each zone), then with the considerable simplification of the system of equations itself, one may obtain good agreement between calculation and experiment. In doing so the results arrived at and the results expected do not match so well, but on the other hand it is possible to make a qualitative analysis of the influence of transfer coefficients on the moisture and temperature potentials. The results obtained through such an approach are well known [10]. Therefore, an efficient method of solution for the linear system with constant transport coefficients plays an important role in solving the nonlinear system of equations. For linear problems formal exact solutions were obtained by a number of workers. Some of these contributions were also summarized in the monographs of Luikov [2, 11] Luikov and Mikhailov [9], Mikhailov and Özisik [12], Shukla [13] and others [14, 15], which deal with basic mathematical tools behind such developments, integral transforms method. Later on, Rossen and Hayakawa [16], Lobo et al. [17], Liu and Cheng [18] noticed that several of the early computational work could be in error, mainly for those results reported at the early times, because of noninclusion of complex roots in such earlier contributions. These authors reported numerical difficulties in computing the complex conjugate characteristic roots, confining their evaluations to one single pair. In view of the limited usefulness of the formal exact solution, Ribeiro et al. [19] proposed an alternative approximate solution to Luikov equations in linear formulation which does not require evaluation of complex eigenvalues.

The present authors have developed a novel technique that provides complete and satisfactory solutions to such system subject to specified initial and surface conditions. Here it is applied to obtain the temperature and moisture distributions during contact drying of a moist porous sheet. Further, the complex characteristic roots are found by a new technique [20] which is a combination of the bisection and Newton–Raphson methods. This technique evaluates simultaneously real as well as a number of pairs of complex conjugate roots. A set of bench mark results is obtained. The previous analytical solutions are critically examined and compared with existing results, and it has been found that earlier results are in error due to non-inclusion of the complex roots not accounted for in such earlier contributions [2, 6, 8, 11–14, 21]. The importance of the present study is: (i) our method of solution has general application to the problems of simultaneous heat and mass transfer in capillary porous bodies, (ii) the methodology for obtaining the real as well as complex roots is quite different from that of Lobo et al. [17] and Liu and Cheng [18] methods, because it evaluates real and a number of pairs of complex conjugate roots, (iii) numerical results obtained by the present technique may serve to check the accuracy of any numerical methods such as finite difference and finite element techniques when applied to solving such types of problems.

2. Problem formulation

The proposed approach is demonstrated for simultaneous heat mass transfer within a porous moist sheet which is in contact with a hot plate. Without loss of generality the problem formulation in dimensionless form can be written as [21]

$$\frac{\partial T(x,\tau)}{\partial \tau} = (1+\varepsilon\, Ko\, Lu\, Pn)\frac{\partial^2 T(x,\tau)}{\partial x^2} - \varepsilon\, Ko\, Lu \frac{\partial^2 \theta(x,\tau)}{\partial x^2} \quad (0 < x < 1, \tau > 0). \tag{1}$$

$$\frac{\partial \theta(x,\tau)}{\partial \tau} = -Lu\, Pn \frac{\partial^2 T(x,\tau)}{\partial x^2} + Lu \frac{\partial^2 \theta(x,\tau)}{\partial x^2} \quad (0 < x < 1, \tau > 0). \tag{2}$$

The initial and boundary conditions for the present study are given as follows:

$$T(x,0) = g_1(x); \quad \theta(x,0) = g_2(x), \quad 0 \leqslant x \leqslant 1. \tag{3}$$

$$\frac{\partial T(0,\tau)}{\partial x} = -Ki_q, \quad \tau > 0 \tag{4}$$

$$\frac{\partial \theta(0,\tau)}{\partial x} - Pn \frac{\partial T(0,\tau)}{\partial x} = 0, \quad \tau > 0 \tag{5}$$

$$\frac{\partial T(1,\tau)}{\partial x} + A_1 T(1,\tau) + B_1 \theta(1,\tau) = \phi_1(\tau), \quad \tau > 0 \tag{6}$$

$$\frac{\partial \theta(1,\tau)}{\partial x} + A_2 \frac{\partial T(1,\tau)}{\partial x} + B_2 \theta(1,\tau) = \phi_2(\tau), \quad \tau > 0. \tag{7}$$

The set of equations (1) and (2) have been solved by Luikov and Mikhailov [9] addressed to various types of boundary conditions. They have also examined a situation where the specific flux of mass varies continuously with time. Boundary conditions (6) and (7) represent still more general cases where the source terms $\phi_j(\tau)$, $(j = 1, 2)$ are certain unknown functions of time, to be determined by the experiment. A_j, B_j $(j = 1, 2)$ are thermophysical coefficients, which are functions of dimensionless transfer coefficients.

The dimensionless variables and the dimensionless thermophysical coefficients are defined as

$x = \dfrac{x^*}{R}$, dimensionless space variable

$\tau = \dfrac{a_q t}{R^2}$, dimensionless time

$T(x,\tau) = \dfrac{v(x^*,\tau) - v_0}{v_s - v_0}$, dimensionless temperature

$\theta(x,\tau) = \dfrac{w_0 - w(x^*,\tau)}{w_0 - w_p}$, dimensionless moisture potential

$Ko = \dfrac{\rho(w_0 - w_p)c_m}{c_q(v_s - v_0)}$, Kossovich number

$Lu = \dfrac{a_m}{a_q}$, Luikov number

$Pn = \dfrac{\delta(v_s - v_0)}{(w_0 - w_p)}$, Posnov number

$Bi_q = \dfrac{\alpha_q R}{\lambda_q}$, Biot number heat transfer

$Bi_m = \dfrac{\alpha_m R}{\lambda_m}$, Biot number for mass transfer

$Ki_q = \dfrac{I_q R}{\lambda_q(v_s - v_0)}$, dimensionless heat flux

where a_q, a_m, etc. are defined in the Nomenclature.

3. Solution procedure

The matrix differential equation is appealing by its close similarity to the differential equation, and in a way, offers the possibility to unify the system given in (1)–(7). Therefore we have the system in the matrix notation as:

$$\frac{\partial \mathbf{Z}(x,\tau)}{\partial \tau} = \mathbf{A}\frac{\partial^2 \mathbf{Z}}{\partial x^2}, \quad 0 < x < 1, \quad \tau > 0 \tag{8}$$

subject to the initial and boundary conditions:

$$\mathbf{Z}_0 = \mathbf{Z}(x,0) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}, \quad 0 \leqslant x \leqslant 1$$

$$\frac{\partial \mathbf{Z}(0,\tau)}{\partial x} = \mathbf{Q} = \begin{pmatrix} -Ki_q \\ -Pn\, Ki_q \end{pmatrix}, \quad \tau > 0 \tag{9}$$

$$\mathbf{N}\frac{\partial \mathbf{Z}(1,\tau)}{\partial x} + \mathbf{M}Z(1,\tau) = \mathbf{H}(\tau), \quad \tau > 0 \tag{10}$$

where

$$\mathbf{Z}(x,\tau) = \begin{pmatrix} T(x,\tau) \\ \theta(x,\tau) \end{pmatrix}$$

$$\mathbf{H}(\tau) = \begin{pmatrix} \phi_1(\tau) \\ \phi_2(\tau) \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 1+\varepsilon\, Ko\, Lu\, Pn & -\varepsilon Ko\, Lu \\ -Lu\, Pn & Lu \end{pmatrix}$$

$$\mathbf{B} = \mathbf{A}^{-1}$$

$$\mathbf{N} = \begin{pmatrix} 1 & 0 \\ A_2 & 1 \end{pmatrix}$$

and

$$\mathbf{M} = \begin{pmatrix} A_1 & B_1 \\ 0 & B_2 \end{pmatrix}.$$

For the boundary condition of the third kind, the thermophysical coefficients assume the following values:

$$A_1 = Bi_q, \quad A_2 = Pn$$

$$B_1 = -(1-\varepsilon)\, Ko\, Lu\, Bi_m, \quad B_2 = Bi_m$$

$$\phi_1(\tau) = Bi_q V_a(\tau) - (1-\varepsilon)\, Ko\, Lu\, Bi_m, \quad \phi_2(\tau) = Bi_m$$

where $V_a(\tau)$ is a dimensionless ambient temperature and matrices **A**, **B**, **M**, **N** and **Q** have their elements which are functions of dimensionless transfer coefficients.

The system under the Laplace transform provides:

$$\frac{d^2\bar{\mathbf{Z}}(x,s)}{dx^2}+\mathbf{B}\bar{\mathbf{Z}}(x,s) = -\mathbf{B}\mathbf{Z}_0(x), \quad (0<x<1, \tau>0) \tag{11}$$

$$\frac{d\bar{\mathbf{Z}}(0,s)}{dx} = \frac{\bar{\mathbf{Q}}}{s} \tag{12}$$

$$\frac{\mathbf{N}\, d\bar{\mathbf{Z}}(1,s)}{dx}+\mathbf{M}\bar{\mathbf{Z}}(1,s) = \bar{\mathbf{H}}(s). \tag{13}$$

Equation (11) is a nonhomogeneous matrix differential equation therefore its solution is formidable. However, if the general solution of the corresponding homogeneous differential equation is known, then the solution of the matrix differential equation (11) can be arrived at by the method of variation of parameters [22]. The method of variation of parameters permits the solution of (11) subject to the conditions (12) and (13) in the form:

$$\bar{\mathbf{Z}}(x,s) = (\cosh\sqrt{\mathbf{B}s}x)\bar{\mathbf{C}}_1(s) + (\sinh\sqrt{\mathbf{B}s}x)\bar{\mathbf{C}}_2(s)+\bar{\mathbf{G}}(x,s)$$

$$\bar{\mathbf{G}}(x,s) = \frac{\sqrt{\mathbf{B}}}{\sqrt{s}}\int^{x}[\sinh\sqrt{Bs}(x-x')]Z_0(x')\,dx'$$

$$\bar{\mathbf{C}}_2(s) = (\mathbf{B}s)^{-1/2}\left[\frac{\bar{\mathbf{Q}}}{s}-\mathbf{G},x(0,s)\right]$$

$$\bar{\mathbf{C}}_1(s) = [\mathbf{N}\sqrt{\mathbf{B}s}\sinh\sqrt{\mathbf{B}s}+\mathbf{M}\cosh\sqrt{\mathbf{B}s}]^{-1}\bar{\mathbf{K}}(s)$$

$$\bar{\mathbf{K}}(s) = \bar{\mathbf{H}}(s)-\mathbf{N}\bar{\mathbf{G}},x(1,s)-\mathbf{M}\bar{\mathbf{G}}(1,s)-(\mathbf{N}\sqrt{\mathbf{B}s}\cosh\sqrt{\mathbf{B}s} +\mathbf{M}\sinh\sqrt{\mathbf{B}s})(\mathbf{B}s)(\mathbf{B}s)^{-1/2}\left(\frac{\mathbf{Q}}{s}-\bar{\mathbf{G}},x(0,s)\right). \tag{14}$$

and $\bar{\mathbf{G}}$, $x(x,s)$ denotes the derivative with respect to x.

In view of Appendix A, equation (14) assumes the form:

$$\bar{\mathbf{Z}}(x,s) = \begin{pmatrix}\bar{T}(x,s)\\ \bar{\theta}(x,s)\end{pmatrix}$$

where

$$\bar{T}(x,s) = \frac{\langle\bar{\phi}_3(s)\bar{Q}_2(s)-\bar{\phi}_4(s)\bar{P}_2(s)-\bar{S}_1(s)\rangle}{\bar{\Delta}(s)}\cosh w_1\sqrt{s}x + \frac{\langle\bar{\phi}_4(s)\bar{P}_1(s)-\bar{\phi}_3(s)\bar{Q}_1(s)-\bar{S}_2(s)\rangle}{\bar{\Delta}(s)}\cosh w_2\sqrt{s}x + \bar{C}_3^*(s)\sinh w_1\sqrt{s}x+\bar{C}_4^*(s)\sinh w_2\sqrt{s}x+\bar{\psi}_1(x,s) \tag{15}$$

$$\bar{\theta}(x,s) = -\frac{1}{\varepsilon Ko}\left[\frac{\langle\bar{\phi}_3(s)\bar{Q}_2(s)-\bar{\phi}_4(s)\bar{P}_2(s)-\bar{S}_1(s)\rangle(1-w_1^2)}{\bar{\Delta}(s)}\times\cosh w_1\sqrt{s}x + \frac{\langle\bar{\phi}_4(s)\bar{P}_1(s)-\bar{\phi}_3(s)\bar{Q}_1(s)-\bar{S}_2(s)\rangle(1-w_2^2)}{\bar{\Delta}(s)}\times\cosh w_2\sqrt{s}x + \bar{C}_3^*(1-w_1^2)\sinh w_1\sqrt{s}x+\bar{C}_4^*(1-w_2^2)\sinh w_2\sqrt{s}x\right] + \bar{\psi}_2(x,s) \tag{16}$$

where $\bar{S}_1(s)$, $\bar{S}_2(s)$, etc. are given in Appendix A.

Employing the expansion theorem and convolution property of the Laplace transform [24], we have the original of $\bar{\mathbf{Z}}(x,s)$ in the form:

$$\mathbf{Z}(x,\tau) = \begin{pmatrix}T(x,\tau)\\ \theta(x,\tau)\end{pmatrix}$$

$$T(x,\tau) = \sum_{j=1}^{2}(-1)^{3-j}\langle W_{3-j}(x,\tau)^*\phi_3(\tau) - Y_{3-j}(x,\tau)^*\phi_4(\tau)\rangle - \sum_{n=1}^{\infty}\langle S_{n1}(u_n)\cos w_1u_nx + S_{n2}(u_n)\cos w_2u_nx\rangle e^{-u_n^2\tau}+(R_1w_1+R_2w_2)x+\psi_1(x,\tau). \tag{17}$$

$$\theta(x,\tau) = -\frac{1}{\varepsilon Ko}\sum_{j=1}^{2}(-1)^{3-j}\langle W_{3-j}(x,\tau)^*\phi_3(\tau) - Y_{3-j}(x,\tau)^*\phi_4(\tau)\rangle(1-w_j^2) - \frac{1}{\varepsilon Ko}\sum_{n=1}^{\infty}\langle S_{n1}(u_n)(1-w_1^2)\cos w_1u_nx + S_{n2}(u_n)(1-w_2^2)\cos w_2u_nx\rangle e^{-u_n^2\tau} - \frac{1}{\varepsilon Ko}\langle R_1w_1(1-w_1^2)+(R_2w_2(1-w_2^2)\rangle x+\psi_2(x,\tau) \tag{18}$$

where S_{n1}, S_{n2}, etc. are given in Appendix B. The u_n $(n = 1,2,\ldots)$ are roots of characteristic equation $E(u_n) = 0$, where

$$E(u_n) = P_{n1}Q_{n2}-P_{n2}Q_{n1}. \tag{19}$$

In view of the initial and boundary conditions, the general expressions obtained for transfer potentials describe a large class of heat mass transfer phenomena including the radiative heat transfer. On specializing transport coefficients involved in the boundary conditions, the solution to numerous specific one-dimensional, time dependent heat and mass diffusion problems encountered in a large variety of applications may be obtained as a special case.

4. Application

In many practical situations without loss of generality [2, 11, 12, 18], the initial distributions and source functions are assumed to be constant. The explicit expressions for transfer potentials in case of the boundary conditions of the third kind at $x = 1$ can easily be written from the general results (17) and (18) after ignoring the intermediate steps as follows:

$$T(x,\tau) = 1-K_1-K_2+(R_1W_1+R_2W_2)x+\sum_{n=1}^{\infty}\sum_{j=1}^{2}(A^*_{nj}+G_{n(4+j)})\cos w_j u_n x\,\mathrm{e}^{-u_n^2\tau} \quad (20)$$

$$\theta(x,\tau) = 1+\frac{1}{\varepsilon Ko}\langle[K_1(1-w_1^2)+K_2(1-w_2^2) -(R_1W_1(1-w_1^2)+(R_2W_2(1-w_2^2))x]\rangle -\frac{1}{\varepsilon Ko}\sum_{n=1}^{\infty}\sum_{j=1}^{2}(A^*_{nj}+G_{n(4+j)})(1-w_j^2)\cos w_j u_n x\,\mathrm{e}^{-u_n^2\tau} \quad (21)$$

where K_1, K_2, etc. are given in Appendix B.

5. Discussion

It can be seen from equation (19) that there is an infinite number of roots $u_1, u_2, \ldots,$ and each subsequent root is greater than the previous one. The roots have been computed with the accuracy of seven decimal places by using a novel technique which evaluates real as well as complex roots and is quite different from the numerical procedure adopted by Lobo et al. [17]. This technique is a combination of the bisection method which determines the real roots and Newton–Raphson method utilising the complex arithmetic determines the complex roots. The numerical procedure chosen in this paper is promising and attractive in the sense that it requires only one input parameter for the evaluation of a number of pairs of complex conjugate roots, whereas Lobo et al. [17] employed a procedure using IMS Library (Zanlyt) to compute the complex roots of the transcendental equation (19). In order to check these results with a different scheme, the Downhill method which evaluates the complex roots of the transcendental equation was employed. Both these methods needed a starting value of the complex eigenvalue as an input parameter. To select a starting value, one has to search for a domain where a pair of complex conjugate roots might exist. This was determined by them employing the method by Ward. Utilizing these methods, they could find only one pair of complex conjugate roots after an exhaustive search. Liu and Cheng [18] employed the procedure adopted by Müller. They also obtained only one pair of complex conjugate roots. In order to study the influence of the inclusion of the complex roots on the dimensionless temperature and moisture distribution and also on the local rate of drying, the following dimensionless parameters as reported in Mikhailov and Shishedjiev [21] and Lobo et al. [17] with some additional values of Bi_{m} are considered:

$$\varepsilon = 0.2, 0.8; \quad Lu = 0.4, \quad Pn = 0.6, \quad Ko = 5.0,$$
$$Ki_q = 0.9, \quad Bi_q = 2.5, \quad Bi_{\mathrm{m}} = 1.0, 2.5, 5.0, 10.0.$$

It is noticed that for each set of the values of the evaporation number ε, the computational technique implemented on IBM compatible PC/AT 386 evaluates 40 real as well as a number of pairs of complex conjugate roots and takes less than 60 s of CPU time which is shown in Table 1. Here it is interesting to note that out of 40 roots, for the dimensionless mass transfer coefficients of interest $Bi_{\mathrm{m}} = 1$, 2.5, 5 and 10, we get, respectively, one, three, three and nine pairs of complex conjugate roots. In order to verify whether these complex roots occurring in Table 1 satisfy the transcendental equation or not, we set $u = \alpha + i\beta$ and $E(u) = E_1(\alpha,\beta)+iE_2(\alpha,\beta)$ where α, β, E_1 and E_2 are real numbers. The numerical values of $E_1(\alpha,\beta)$ and $E_2(\alpha,\beta)$ obtained by this technique with some additional complex roots by extending the domain in the u-complex plane are shown in Table 2. The numerical values thus evaluated may be considered to be zero within the accuracy of our computations showing that these are the roots of the transcendental equation $E(u) = 0$. Figures 1 and 2 exhibit the influence of the inclusion of the complex roots on the temperature and moisture distribution at various dimensionless times $\tau = 0.05$, 0.2, 0.4, 1.6. The dotted curves denote the results corresponding to the real roots only whereas the continuous curves represent the results based on the use of both the real and the complex roots. Obviously the contribution of complex roots is more pronounced for early times and compares well with that of Lobo et al. [17] where they have considered only one pair of complex conjugate roots. The effect of inclusion of complex root on the local rate of drying is also depicted in Fig. 3. It is noticed that for early times, it has significant influence. Similar behaviour is also observed in the case of $\varepsilon = 0.8$. The influence of the inclusion of the complex roots for the value of $\varepsilon = 0.8$ on the temperature and moisture distribution are shown in Figs 4 and 5 and it is found that here also for early times results are significant and do not compare with that of Mikhailov and Shishedjiev [21] because they have overlooked the inclusion of complex roots in their solution. The influence of other dimensionless parameters on transfer potentials and rate of drying has also been studied (not shown here) and a similar behaviour is found. It is therefore essential to include the complex roots in order to get a qualitatively true picture of the temperature and moisture distributions, their average values and the local and average rate of drying. This shows that an analytic solution of the Luikov system of coupled heat and moisture diffusion problems addressed to the linear type of boundary conditions have all to be reviewed since it seems that under certain conditions the transcendental equations have complex roots which are not included in the original computations.

6. Conclusion

The solutions for the dimensionless temperature and moisture distribution obtained by the matrix calculus

Table 1
Characteristic roots of the transcendental equation $P1(u)*Q2(u)-P2(u)*Q1(u)=0$ for different values of Bi_m ($\varepsilon=0.2, K_0=5.0, P_n=0.6, L_u=0.4, Bi_q=2.5$)

$Bi_m = 1.00$	$Bi_m = 2.50$	$Bi_m = 5.00$	$Bi_m = 10.00$
0.5151887	0.6383109	0.6936044	0.7233486
1.3430031	1.6831976	1.8072700	1.9267839
1.7917267	±0.1629899	±0.2748241	±0.3014619
3.5245321	3.6156235	3.7075735	3.7824414
4.2275905	4.3637485	4.7352348	4.9937089
5.1680663	5.1139688	4.9146338	±0.4187257
6.8976957	6.9199545	6.9494397	6.9883604
7.6257266	7.7105249	7.8775266	8.2851406
8.5891456	8.5575339	8.4872077	±0.2457756
10.3113469	10.3144293	10.3189025	10.3259476
11.1611546	11.2274913	11.3495908	11.7202218
12.0155292	11.9922605	11.9431618	±0.1316761
13.7348880	13.7320374	13.7276803	13.7201695
14.7438127	14.7991141	14.9001927	15.2010663
15.4435756	15.4242334	15.3830218	±0.1162802
17.1618068	17.1570431	17.1495146	17.1357782
18.3478767	18.3963569	18.4879401	18.7010367
18.8722524	18.8543561	18.8135409	±0.1439593
20.5901615	20.5848723	20.5763219	20.5600985
21.9636537	22.0097061	22.1161275	22.2112198
22.3009336	22.2813460	22.2189079	±0.1606051
24.0192306	24.0139282	24.0052171	23.9882088
25.5882808	25.6696243	25.6889970	25.7276686
25.7277073	±0.0166561	±0.0907089	±0.1477532
27.4486985	27.4435705	27.4350422	27.4180167
29.1859147	29.1962785	29.2135836	29.2482969
±0.0140942	±0.0387784	±0.0566025	±0.0693325
30.8784086	30.8735117	30.8652866	30.8485602
32.5957168	32.5977951	32.6007201	32.6050999
32.8349623	32.8515793	32.8799485	32.9386607
34.3082753	34.3036100	34.2957052	34.2793596
36.0243953	36.0233019	36.0216468	36.0188426
36.4672646	36.4854055	36.5156707	36.5764718
37.7382473	37.7337871	37.7261656	37.7101408
39.4539993	39.4519222	39.4486660	39.4428176
40.1002076	40.1179914	40.1476870	40.2074841
41.1682908	41.1639922	41.1565783	41.1406936
42.8838960	42.8814472	42.8775285	42.8702386
43.7340465	43.7511237	43.7797654	43.8379630
44.5983805	44.5941798	44.5868495	44.5707524

±: Imaginary part of complex root with real part just above it

form sufficiently general expressions from which solutions to many specific one-dimensional, time dependent heat and mass diffusion problems encountered in a large variety of applications may be obtained as a particular case. The roots finding procedure is attractive in the sense that it evaluates real as well as a number of pairs of complex conjugate roots of the transcendental equation. The inclusion of the complex eigenvalues in the analysis is felt to be of practical importance because the complex roots have been overlooked in the earlier contributions.

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Table 2
Complex roots of the transcendental equation $P1(u)*Q2(u)-P2(u)*Q1(u)=0$ for different values of Bi_{m} ($\varepsilon=0.2, K_0=5.0, L_u=0.4, P_n=0.6, Bi_q=2.5$)

Bi_{m}	Complex root $=\alpha\pm i\beta$		$E(u)=E_1(\alpha,\beta)\pm iE_2(\alpha,\beta)$	
	α	β	E_1	E_2
1.00	29.1859147	0.0140942	0.000000000000	0.000000000000
1.00	149.2260709	0.0047647	−0.000000000001	0.000000000000
2.50	1.6831976	0.1629899	0.000000000642	0.000000000562
2.50	25.6696243	0.0166561	−0.000000000107	−0.000000000039
2.50	29.1962785	0.0387784	−0.000000004783	−0.000000001351
2.50	58.3026230	0.0229925	−0.000000000000	−0.000000000000
2.50	149.2280818	0.0093330	−0.000000000000	−0.000000000007
5.00	1.8072700	0.2748241	0.000000000000	0.000000000000
5.00	25.6889970	0.0907089	0.000000000000	0.000000000002
5.00	29.2135836	0.0566025	−0.000000000001	0.000000000001
5.00	58.3111978	0.0425494	−0.000000000002	0.000000000001
5.00	149.2314335	0.0132844	0.000000000000	−0.000000000000
5.00	298.4402481	0.0066737	0.000000000000	0.000000000000
10.00	1.9267839	0.3014619	−0.000000000001	−0.000000000000
10.00	4.9937089	0.4187257	−0.000000000000	0.000000000000
10.00	8.2851406	0.2457756	−0.000000000002	0.000000000000
10.00	11.7202218	0.1316761	0.000000000027	0.000000000017
10.00	15.2010663	0.1162802	−0.000000000000	−0.000000000000
10.00	18.7010367	0.1439593	−0.000000000000	0.000000000000
10.00	22.2112198	0.1606051	0.000000000000	0.000000000001
10.00	25.7276686	0.1477532	0.000000000351	−0.000000000287
10.00	29.2482969	0.0693325	0.000000000000	−0.000000000000
10.00	58.3283456	0.0627075	0.000000003141	−0.000000000952
10.00	87.4460365	0.0394649	0.000000000000	−0.000000000000
10.00	149.2381375	0.0169651	0.000000000188	−0.000000000175
10.00	178.3708748	0.0209960	−0.000000000000	0.000000000000
10.00	298.4435985	0.0131457	0.000000000000	0.000000000000

Appendix A

The matrix **B** is diagonalizable and it can be expressed in terms of characteristic roots and characteristic vectors. In the light of this, **B** can be written as [23]

$$\mathbf{B}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

where **D** is a diagonal matrix of order 2×2 containing characteristic roots λ_1 and λ_2.

$$\lambda_1=w_1^2=\frac{1}{2}\left\{\left(1+\varepsilon\,Ko\,Pn+\frac{1}{Lu}\right)+\left[\left(1+\varepsilon\,Ko\,Pn+\frac{1}{Lu}\right)^2-\frac{4}{Lu}\right]^{1/2}\right\}$$

$$\lambda_2=w_2^2=\frac{1}{2}\left\{\left(1+\varepsilon\,Ko\,Pn+\frac{1}{Lu}\right)-\left[\left(1+\varepsilon\,Ko\,Pn+\frac{1}{Lu}\right)^2-\frac{4}{Lu}\right]^{1/2}\right\}$$

and **P** is the matrix of eigenvectors obtained from **B** as

$$\mathbf{P}=\begin{pmatrix}1 & 1\\ -\dfrac{(1-w_1^2)}{\varepsilon\,Ko} & -\dfrac{(1-w_2^2)}{\varepsilon\,Ko}\end{pmatrix}$$

and

$$\mathbf{P}^{-1}=\begin{bmatrix}-\dfrac{(1-w_2^2)}{\varepsilon\,Ko} & -1\\ \dfrac{(1-w_1^2)}{\varepsilon\,Ko} & 1\end{bmatrix}\Bigg/(w_2^2-w_1^2)(\varepsilon\,Ko)^{-1}.$$

In view of this we can write

$$\mathbf{B}^2=\mathbf{B}\cdot\mathbf{B}=\mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}$$

and

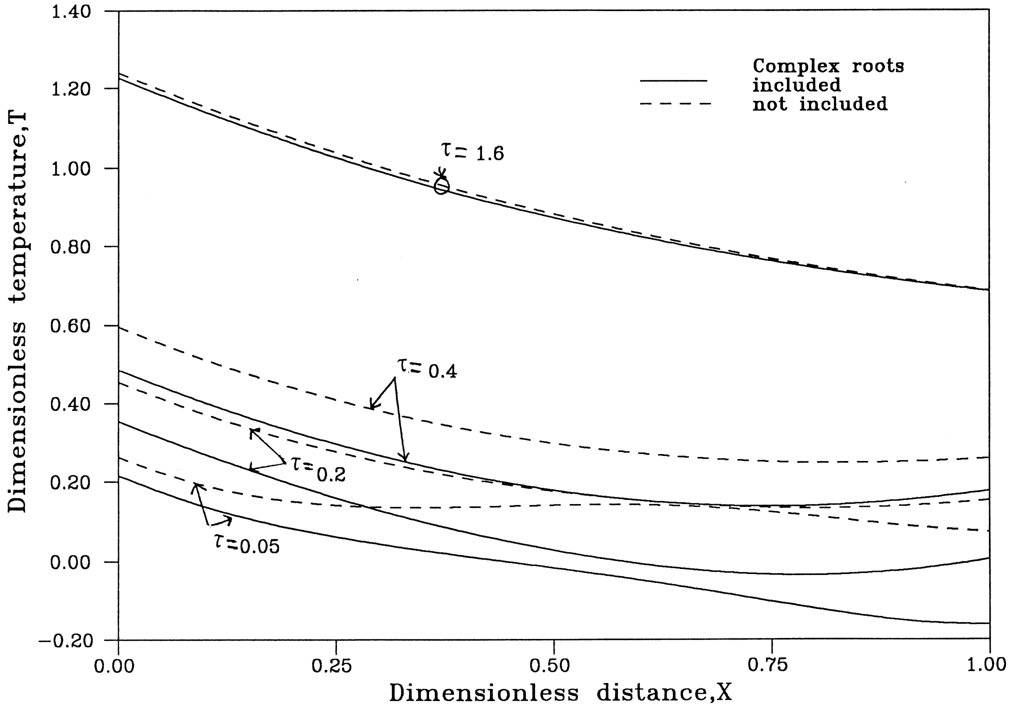

Fig. 1. Effect of inclusion of the complex roots on dimensionless temperature vs. dimensionless distance ($\varepsilon = 0.2$; $Ko = 5.0$; $Lu = 0.4$; $Pn = 0.6$; $Bi_q = 2.5$; $Bi_{\mathrm{m}} = 2.5$; $Ki_q = 0.9$).

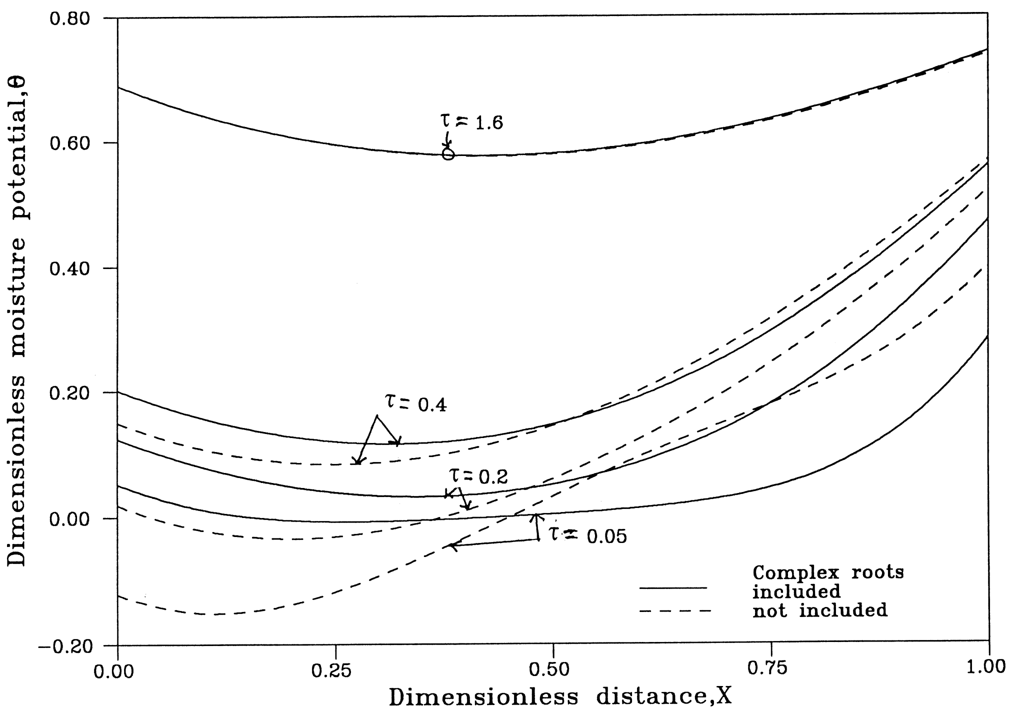

Fig. 2. Effect of inclusion of the complex roots on dimensionless moisture potential vs. dimensionless distance ($\varepsilon = 0.2$; $Ko = 5.0$; $Lu = 0.4$; $Pn = 0.6$; $Bi_q = 2.5$; $Bi_{\mathrm{m}} = 2.5$; $Ki_q = 0.9$).

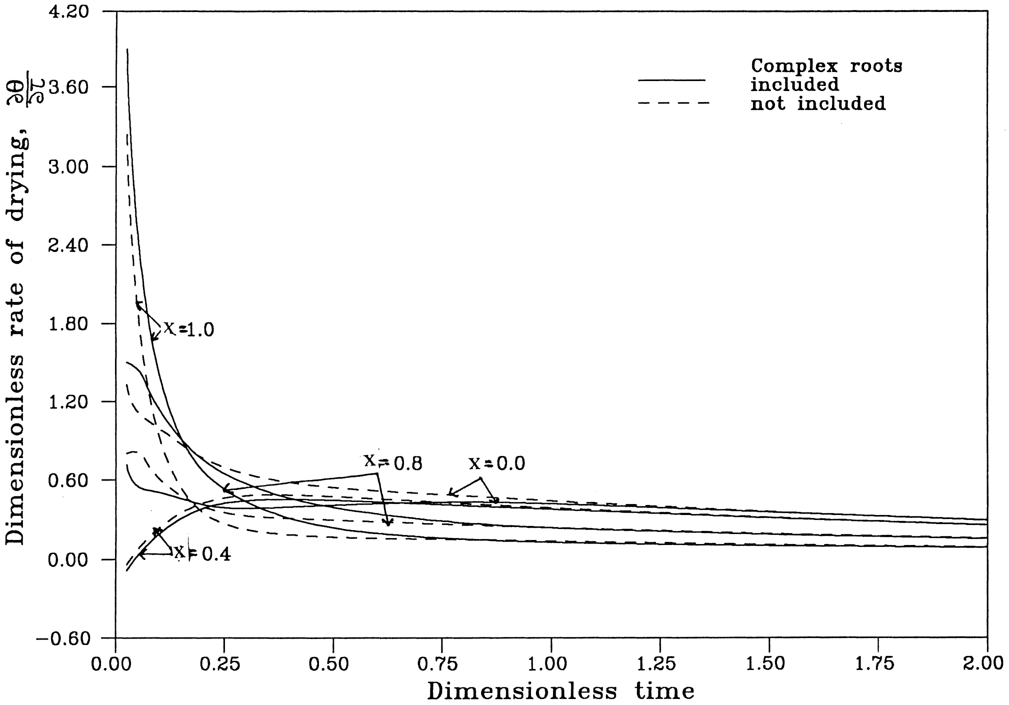

Fig. 3. Effect of inclusion of the complex roots on dimensionless rate of drying vs. dimensionless time ($\varepsilon = 0.2$; $Ko = 5.0$; $Lu = 0.4$; $Pn = 0.6$; $Bi_q = 2.5$; $Bi_m = 2.5$; $Ki_q = 0.9$).

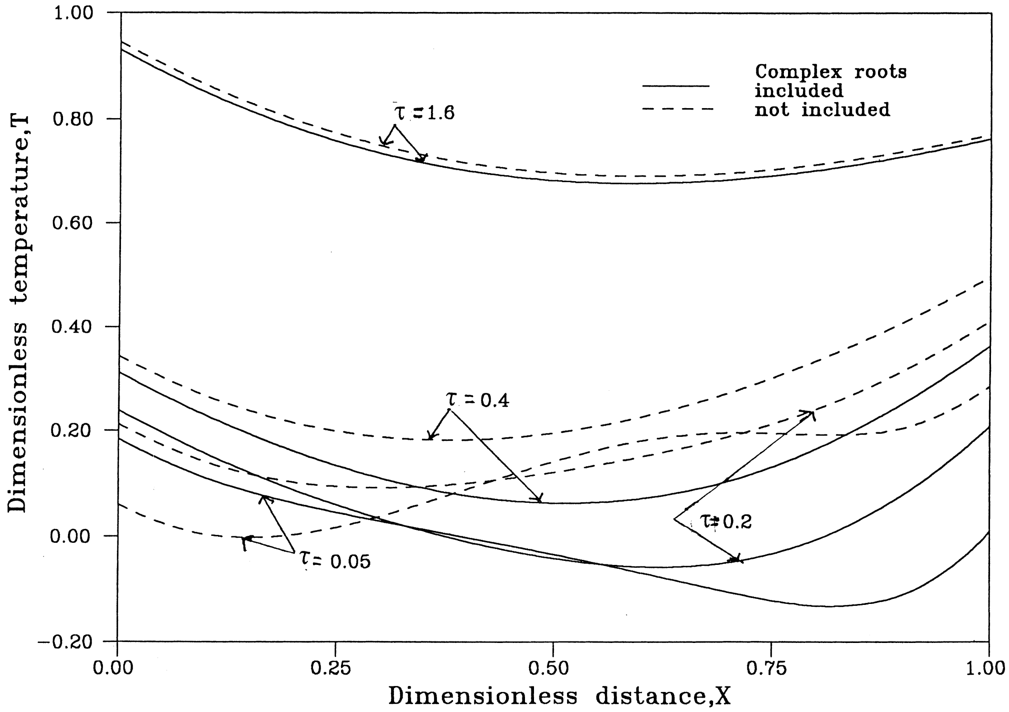

Fig. 4. Effect of inclusion of the complex roots on dimensionless temperature vs. dimensionless distance ($\varepsilon = 0.8$; $Ko = 5.0$; $Lu = 0.4$; $Pn = 0.6$; $Bi_q = 2.5$; $Bi_m = 2.5$; $Ki_q = 0.9$).

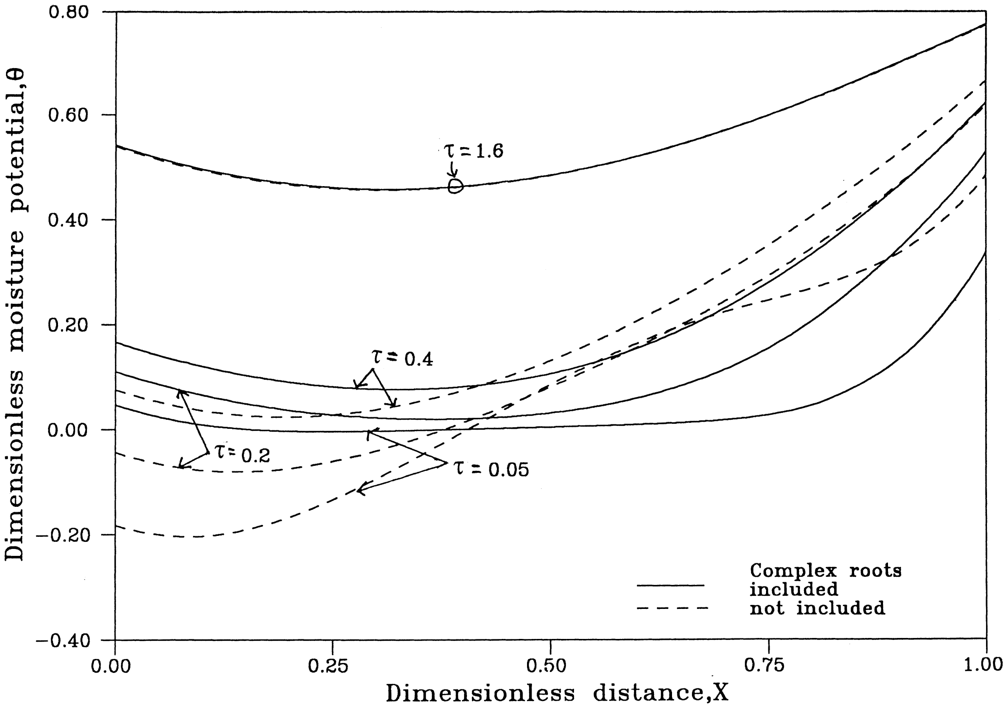

Fig. 5. Effect of inclusion of the complex roots on dimensionless moisture potential vs. dimensionless distance ($\varepsilon = 0.8$; $Ko = 5.0$; $Lu = 0.4$; $Pn = 0.6$; $Bi_q = 2.5$; $Bi_m = 2.5$; $Ki_q = 0.9$).

$$\mathbf{B}^3 = \mathbf{B}^2 \cdot \mathbf{B} = \mathbf{PD}^3\mathbf{P}^{-1}.$$

The above can be extended to yield

$$f(\mathbf{B}) = \sum_{n=1}^{\infty} \alpha_n \mathbf{B}^n = \mathbf{P}\left(\sum_{n=1}^{\infty} \alpha_n \mathbf{D}^n\right)\mathbf{P}^{-1} = \mathbf{P}\begin{pmatrix} f(\lambda_1) & 0 \\ 0 & f(\lambda_2) \end{pmatrix}\mathbf{P}^{-1}. \quad \text{(A1)}$$

The above expression provides an efficient way to evaluate a function of diagonalizable matrix in terms of eigenvalues and eigenvectors. Further, the eigenvalues of **B** are real and positive, we can write

$$\sqrt{\mathbf{B}} = \mathbf{P}\begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}\mathbf{P}^{-1}$$

and

$$\mathbf{B}^{-1/2} = \mathbf{P}\begin{pmatrix} w_1^{-1} & 0 \\ 0 & w_2^{-1} \end{pmatrix}\mathbf{P}^{-1} \quad \text{(A2)}$$

and in view of (A1), we have

$$f(\sqrt{\mathbf{B}}) = \mathbf{P}\begin{pmatrix} f(w_1) & 0 \\ 0 & f(w_2) \end{pmatrix}\mathbf{P}^{-1}, \text{ etc.} \quad \text{(A3)}$$

For instance

$$\sinh\sqrt{\mathbf{B}s} = \mathbf{P}\begin{pmatrix} \sinh w_1\sqrt{s} & 0 \\ & \sinh w_2\sqrt{s} \end{pmatrix}\mathbf{P}^{-1}, \text{ etc.} \quad \text{(A4)}$$

Utilizing these concepts to the solution in the matrix form given in (14), we get the result mentioned in (15) and (16), where $\bar{S}_1$, $\bar{S}_2$, etc. are obtained in the form:

$$\bar{S}_1(s) = \bar{C}_3^*(s)\bar{G}_3(s) + \bar{C}_4^*(s)\bar{G}_4(s)$$
$$\bar{S}_2(s) = \bar{C}_3^*(s)\bar{G}_1(s) + \bar{C}_4^*(s)\bar{G}_2(s)$$
$$\bar{G}_1(s) = \bar{Q}_3(s)\bar{P}_1(s) + \bar{P}_3(s)\bar{Q}_1(s)$$
$$\bar{G}_2(s) = \bar{Q}_4(s)\bar{P}_1(s) - \bar{P}_4(s)\bar{Q}_1(s)$$
$$\bar{G}_3(s) = \bar{P}_3(s)\bar{Q}_2(s) - \bar{P}_2(s)\bar{Q}_3(s)$$
$$\bar{G}_4(s) = \bar{P}_4(s)\bar{Q}_2(s) - \bar{P}_2(s)\bar{Q}_4(s)$$

$$\bar{\psi}_1(x,s) = \bar{r}_1(s)\int^x \bar{\psi}(x's)\sinh w_1\sqrt{s}(x-x')\,\mathrm{d}x' + \bar{r}_2(s)\int^x \bar{\psi}(x's)\sinh w_2\sqrt{s}(x-x')\,\mathrm{d}x'$$

$$\bar{\psi}_2(x,s) = \frac{(\Delta_x - s)\bar{\psi}_1(x,s) + (\varepsilon Ko\, g_2 + g_1)}{\varepsilon Ko\, s}$$

$$\bar{\psi}(x,s) = -\Delta_x(\varepsilon Ko\, g_2 + g_1) + \frac{sg_1}{Lu}$$

$$\bar{\phi}_3(s) = [\bar{\psi}_1(s) - (\bar{\psi}_{1,x}(1,s) + A_1\bar{\psi}_1(1,s) + B_2\bar{\psi}_2(1,s))]$$
$$\bar{\phi}_4(s) = [\bar{\psi}_2(s) - (\bar{\psi}_{2,x}(1,s) + A_2\bar{\psi}_{1,x}(1,s) + B_2\bar{\psi}_2(1,s))]$$
$$\bar{r}_1(s) = \frac{1}{w_1(w_2^2 - w_1^2)s\sqrt{s}}$$
$$\bar{r}_2(s) = \frac{1}{w_2(w_2^2 - w_1^2)s\sqrt{s}}$$
$$\Delta_x = \frac{\partial^2}{\partial x^2}$$
$$R_1(s) = \frac{Ki_q(1 - w_2^2 + \varepsilon\, Ko\, Pn)}{w_1(w_2^2 - w_1^2)}$$
$$R_2(s) = \frac{Ki_q(1 - w_1^2 + \varepsilon\, Ko\, Pn)}{w_1(w_1^2 - w_2^2)}$$
$$\bar{P}_j(s) = w_j\sqrt{s}\sinh w_j\sqrt{s} + M_1^{(j)}\cosh w_j\sqrt{s}$$
$$\bar{Q}_j(s) = M_2^{(j)} w_j\sqrt{s}\sinh w_j\sqrt{s} + M_3^{(j)}\cosh w_j\sqrt{s}$$
$$\bar{P}_{j+2}(s) = w_j\sqrt{s}\cosh w_j\sqrt{s} + M_1^{(j)}\sinh w_j\sqrt{s}$$
$$\bar{Q}_{j+2}(s) = M_2^{(j)} w_j\sqrt{s}\cosh w_j\sqrt{s} + M_3^{(j)}\sinh w_j\sqrt{s}$$
$$M_1^{(j)} = A_1 - B_1\frac{(1-w_j^2)}{\varepsilon\, Ko}$$
$$M_2^{(j)} = A_2 - \frac{(1-w_j^2)}{\varepsilon\, Ko}$$
$$M_3^{(j)} = -B_2\frac{(1-w_j^2)}{\varepsilon\, Ko} \quad (j = 1, 2)$$
$$C_3^*(s) = -\frac{R_1}{s\sqrt{s}}$$

and

$$C_4^*(s) = \frac{R_2}{s\sqrt{s}}.$$

Appendix B

The elements of $\bar{\mathbf{Z}}(x,s)$ contain the terms $\bar{\phi}_3(s)$, $\bar{\phi}_4(s)$, the true nature of which is not yet defined. Therefore, in order to determine the original of $\bar{\mathbf{Z}}(x,s)$, one will require the expansion theorem where the expression contains all the well-defined terms such as $\bar{P}_j$, $\bar{Q}_j$, etc. and the convolution theorem for the terms $\bar{\phi}_3(s)$, $\bar{\phi}_4(s)$.

Employing these theorems [24], we obtain the original of $\bar{\mathbf{Z}}(x,s)$ in the form

$$\mathbf{Z}(x,\tau) = \begin{pmatrix} T(x,\tau) \\ \theta(x,\tau) \end{pmatrix} \tag{B1}$$

where the transfer potentials $T(x,\tau)$ and $\theta(x,\tau)$ are given by (15) and (16) with

$$S_{n1} = \frac{R_1 G_{n3}(u_n) + R_2 G_{n4}(u_n)}{f_n}$$
$$S_{n2} = \frac{R_1 G_{n1}(u_n) + R_2 G_{n2}(u_n)}{f_n}$$
$$W_{3-j}(x,\tau)*\phi_3(\tau) = \sum_{n=1}^{\infty}\frac{Q_{n(3-j)}\cos w_j u_n x}{f_n} \times \int_0^{\tau} e^{-u_n^2\tau'}\phi_3(\tau-\tau')\,d\tau'$$
$$Y_{3-j}(x,\tau)*\phi_4(\tau) = \sum_{n=1}^{\infty}\frac{P_{n(3-j)}\cos w_j u_n x}{f_n} \times \int_0^{\tau} e^{-u_n^2\tau'}\phi_4(\tau-\tau')\,d\tau'$$
$$P_{nj} = M_1^{(j)}\cos w_j u_n - w_j u_n \sin w_j u_n$$
$$Q_{nj} = M_3^{(j)}\cos w_j u_n - M_2^{(j)} w_j u_n \sin w_j u_n$$
$$P_{n(j+2)} = w_j u_n \cos w_j u_n + M_1^{(j)}\sin w_j u_n$$
$$Q_{n(j+2)} = M_2^{(j)} w_j u_n \cos w_j u_n + M_3^{(j)}\sin w_j u_n$$
$$f_n = P_1^*(u_n)Q_2(u_n) + P_1(u_n)Q_2^*(u_n) - P_2^*(u_n)Q_1(u_n) - P_2(u_n)Q_2^*(u_n)$$
$$P_{nj}^* = \frac{w_j}{2u_n}[(M_3^{(j)} + M_2^{(j)})\sin w_j u_n + M_2^{(j)} w_j u_n \cos w_j u_n] \tag{B2}$$

and u_n $(n = 1, 2, 3, \ldots)$ are the roots of the characteristic equation

$$P_1(u)Q_2(u) - P_2(u)Q_1(u) = 0. \tag{B3}$$

As a particular case when energy and mass transfer takes place according to the convective law and the initial temperature and moisture distributions are constant then the source terms and aggregate of the dimensionless thermophysical coefficients assume the form:

$$\bar{\mathbf{Z}}(x,0) = \mathbf{0}, \quad \phi_1(\tau) = \phi_1^0 = Bi_q - (1-\varepsilon)\, Ko\, Lu\, Bi_{\mathrm{m}}$$
$$\phi_2(\tau) = \phi_2^0 = Bi_{\mathrm{m}}, \quad A_1 = Bi_q, \quad A_2 = -Pn$$
$$B_1 = -(1-\varepsilon)\, Ko\, Lu\, Bi_{\mathrm{m}}, \quad B_2 = Bi_{\mathrm{m}}.$$

Making use of these values in the general results (17) and (18) we get after a series of algebraic manipulation, the results given in (20) and (21) where

$$K_1 = \frac{R_1\langle(1+M_1^{(1)})M_3^{(2)} - (M_2^{(1)} + M_3^{(2)})M_1^{(2)}\rangle w_1 + R_2\langle(1+M_1^{(2)})M_3^{(1)} - (M_2^{(2)} + M_3^{(2)})M_1^{(2)}\rangle w_2}{(M_1^{(1)}M_3^{(2)} - M_1^{(2)}M_3^{(2)})}$$
$$K_2 = \frac{R_1\langle(M_2^{(1)} + M_3^{(1)})M_1^{(1)} - (1+M_1^{(1)})M_3^{(1)}\rangle w_1 + R_2\langle(M_2^{(2)} + M_3^{(2)})M_1^{(1)} - (1+M_1^{(2)})M_3^{(1)}\rangle w_2}{(M_1^{(1)}M_3^{(2)} - M_1^{(2)}M_3^{(1)})}$$
$$A_{nj}^* = (-1)^j\frac{[\phi_1^0 Q_{n(3-j)} - \phi_2^0 P_{n(3-j)}]}{u_n^2 f_n}, \quad (j = 1, 2)$$
$$G_{n5} = \frac{R_1(P_{n3}Q_{n2} - P_{n2}Q_{n3}) - R_2(P_{n4}Q_{n2} - P_{n2}Q_{n4})}{u_n^3 f_n} \tag{B4}$$

and

$$G_{n6} = \frac{R_1(Q_{n3}P_{n1} - P_{n3}Q_{n1}) - R_2(P_{n1}Q_{n4} - P_{n4}Q_{n1})}{u_n^3 f_n}.$$

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